Basic Filters – Part 2

(1-Pole High Pass Filter)

Numerical Solutions to

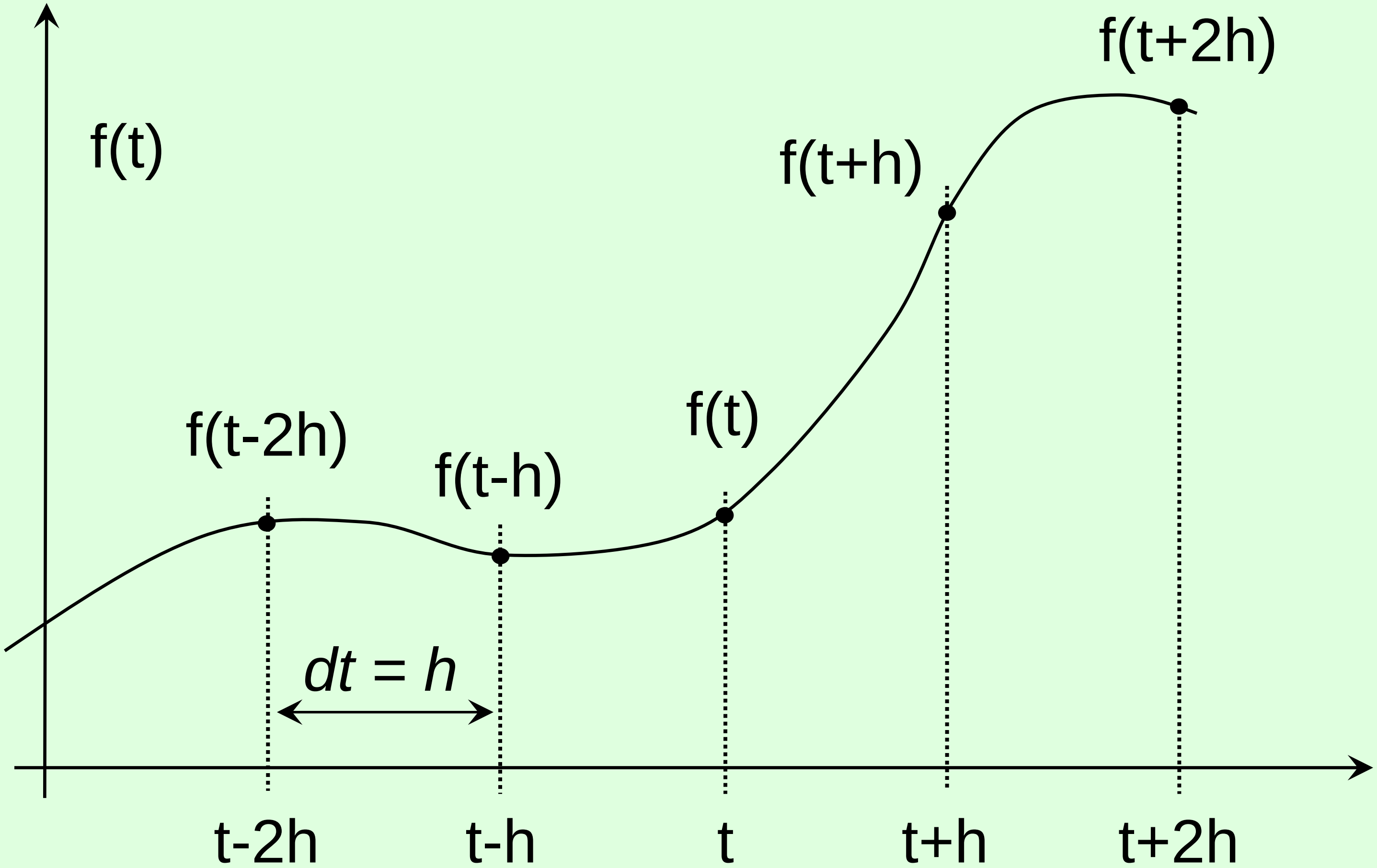
Differential Equations

(Finite Difference Method)

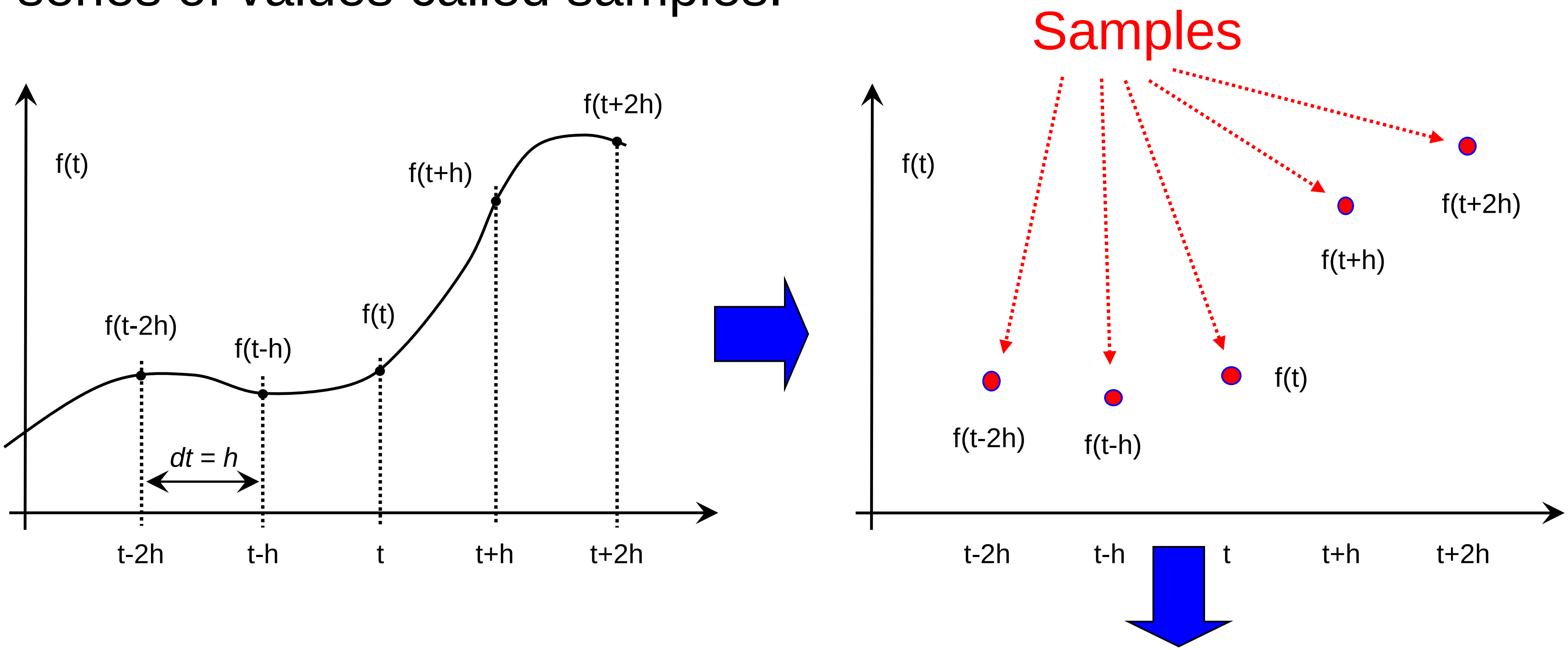
by George Lungu

<excelunusual.com>

In order to numerically model a time dependant process we first need to sample functions at discrete intervals ***dt = h***



Therefore a continuous function will be replaced with a discrete series of values called samples.

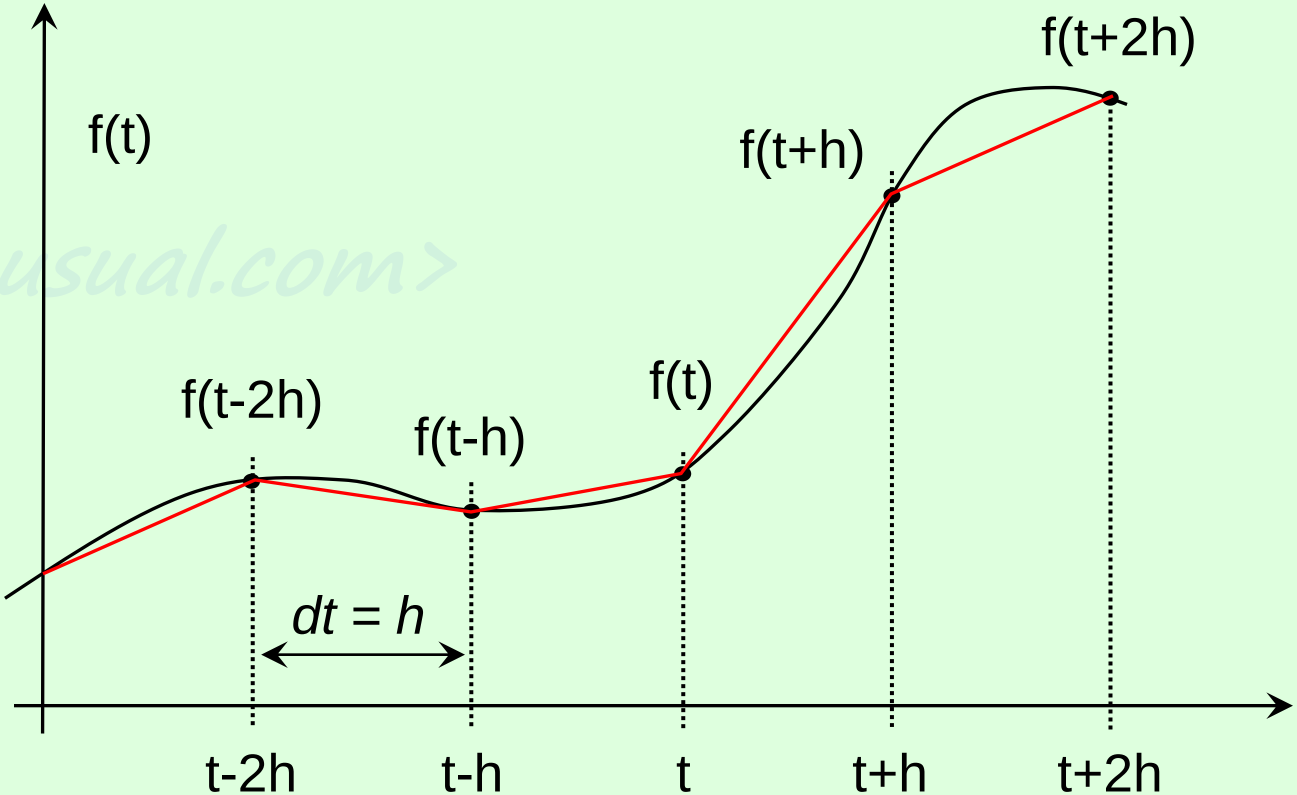


Te computer will keep this information as a 2-column table of numbers

time	F(t)
0	f(0)
h	f(h)
2h	f(2h)
3h	f(3h)
4h	f(4h)

It is just like a movie. In order to record a movie it's enough to store about 30 to 50 still snapshots every second.

So which is the "density" of samples needed in our computerized numerical simulation?

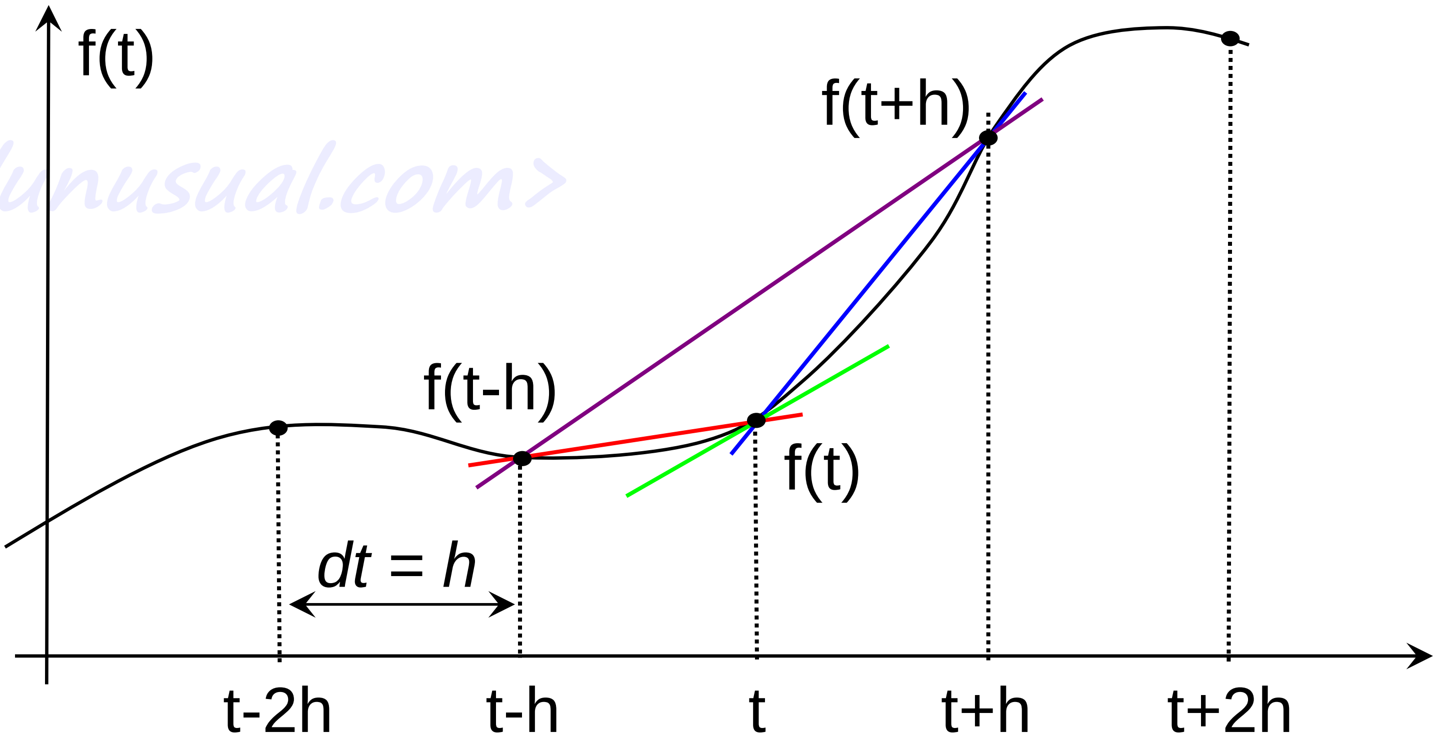


One answer would be: between each two consecutive samples the function should be not deviate much from a straight line

Most of the physical processes can be described by differential equations. We need to find a way of expressing derivatives in an approximate but easy way.

The ***"finite difference method"*** *comes to help:*

This method approximates the tangent to the curve in point "t" (green curve) with something easy, much more convenient in calculation – the secant to the curve (there are three options available around point "t")



First derivative approximations

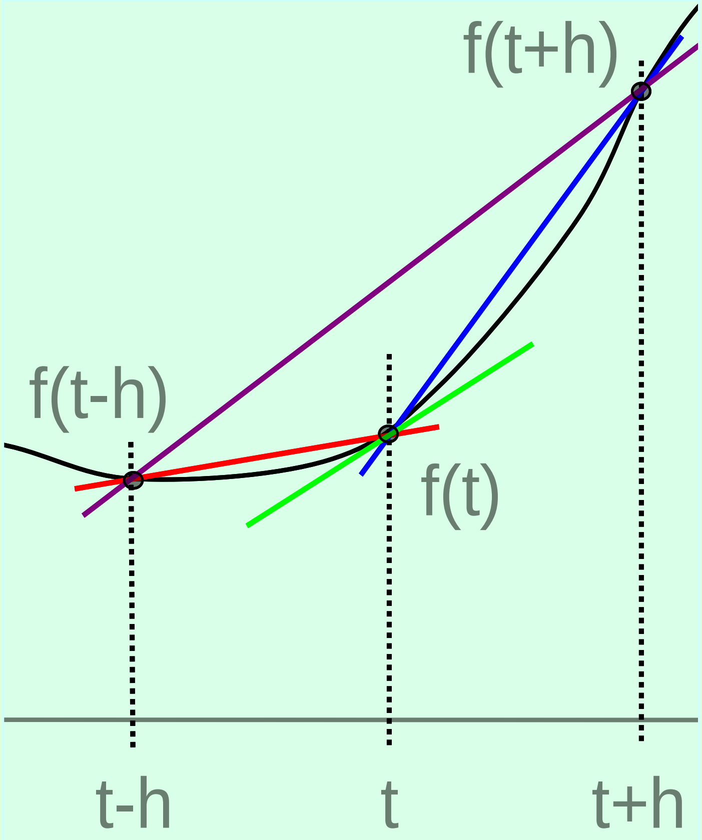
Definition: $$\frac{df(t)}{dt} = \lim_{dt \to 0}\left[\frac{f(t+dt) - f(t)}{dt}\right]$$

Forward estimate: $$\frac{df(t)}{dt} \approx \frac{f(t+h) - f(t)}{h}$$

Backward estimate: $$\frac{df(t)}{dt} \approx \frac{f(t) - f(t-h)}{h}$$

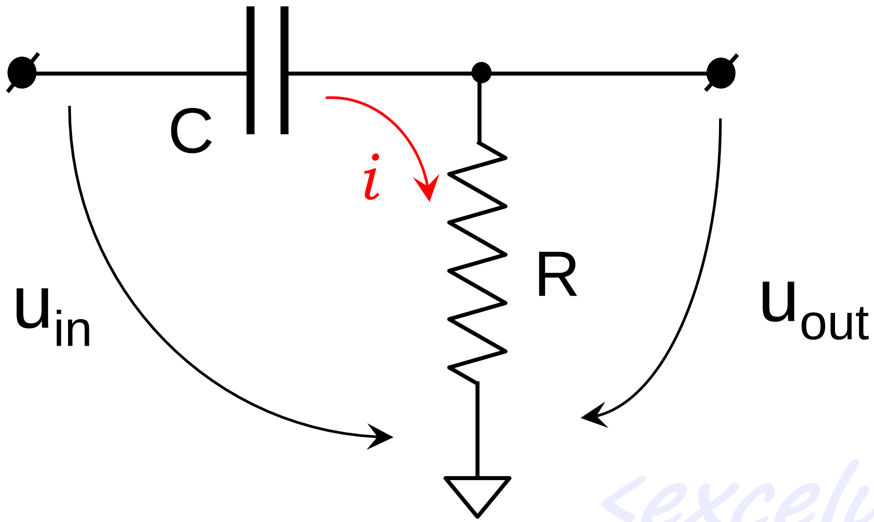
Central estimate: $$\frac{df(t)}{dt} \approx \frac{f(t+h) - f(t-h)}{2 \cdot h}$$

It is obvious that denser sampling (smaller h) leads to better precision



Pay attention to colors!

So how do we use all of this to numerically model a High Pass Filter?



From Ohm's law: $u_{out}(t) = R \cdot i$

From the definition of current intensity: $i = \dfrac{dq}{dt}$

From the definition of capacitance: $C = \dfrac{dq}{d(u_{in} - u_{out})}$

Let’s combine these simple equations:

$$\begin{cases} u_{out}(t) = R \cdot \dfrac{dq}{dt} \\ dq = C \cdot (du_{in} - du_{out}) \end{cases} \quad =>$$

$$u_{out}(t) = R \cdot C \cdot \left(\frac{du_{in}}{dt} - \frac{du_{out}}{dt} \right) \quad =>$$

Using the backward estimate of the derivative
(also use the notation *dt* = *h*)

$$h \cdot u_{out}(t) = R \cdot C \cdot \left(u_{in}(t) - u_{in}(t-h) - u_{out}(t) + u_{out}(t-h) \right) \quad =>$$

$$u_{out}(t) = \frac{u_{out}(t-h) + u_{in}(t) - u_{in}(t-h)}{1 + \frac{h}{R \cdot C}}$$

h - the step size and we choose this constant to be not too large (typically 1-5% of the signal period)

$u_{in}(t)$ - the input signal at moment “t”, we also choose this one as initial condition

$R \cdot C$ - a constant which is equal to the product between the resistance and the capacitance

$u_{out}(t-h)$ - the previous value of the output and it has been already calculated for the previous time step (to start the iteration, $u_{out}(0)$ is chosen as a constant – called an initial condition)

How is this programmed in Excel?

Open a new workbook, select the first worksheet (lower left bottom) and rename it “HPF_backward”

Parameter area:

Cell A4: RC   Cell B4: 0.3

Cell A5: dt   Cell B5: 0.05

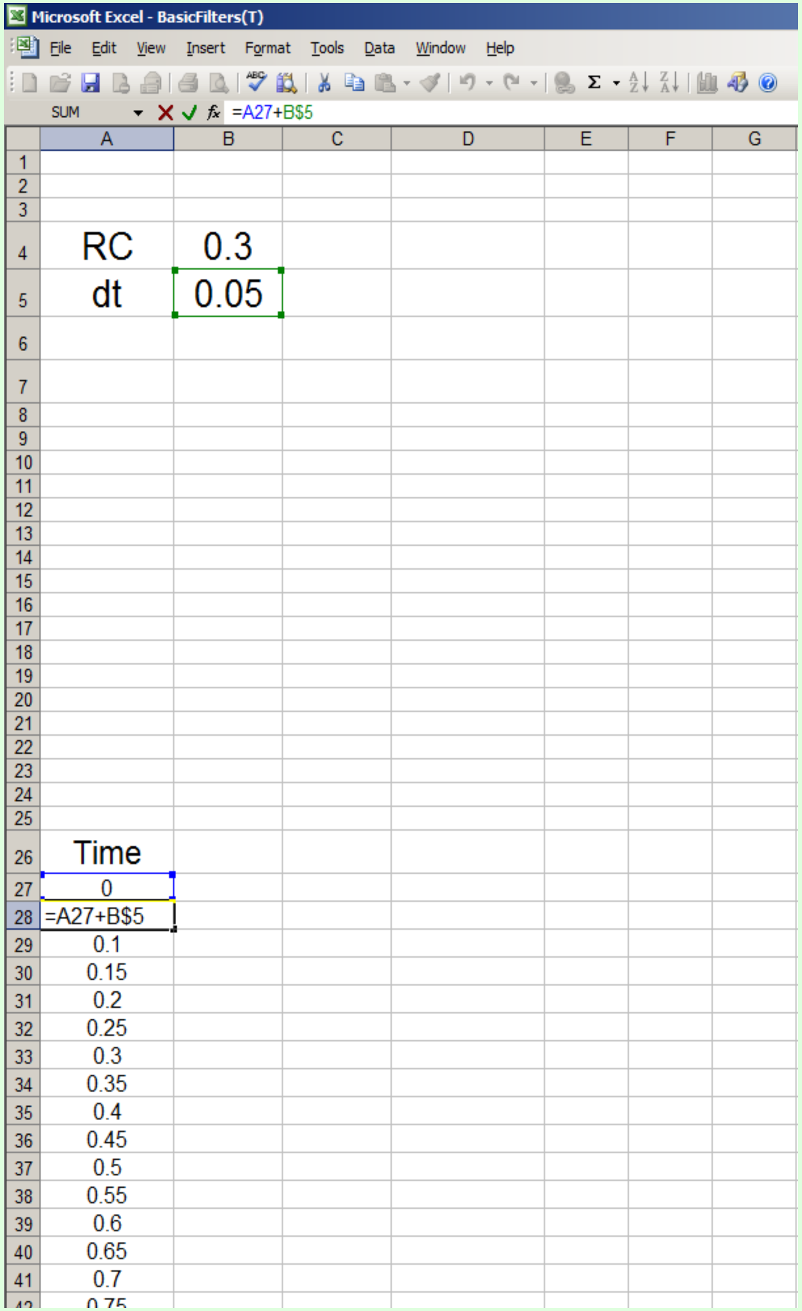
Time column:

Cell A26: Time (a label)

Cell A27: 0

Cell A28: =A27+B$5

Copy A28 down to A800
(this will generate an increasing time series on column A)



Create a rectangular periodic input signal and plot it:

Input signal column:

Cell B26: u_in (a label)

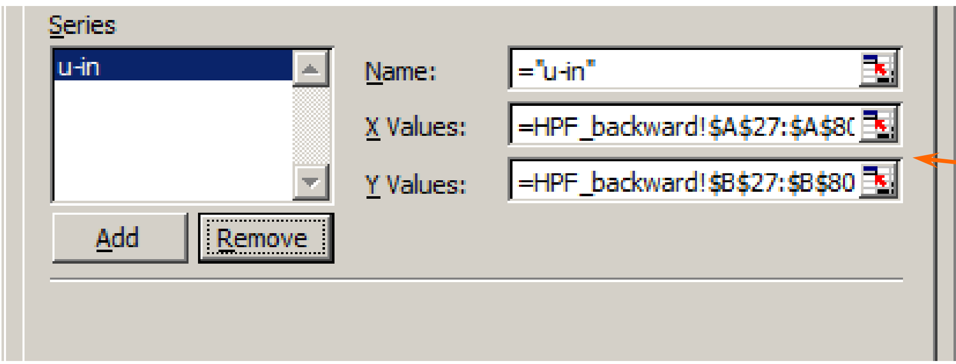
Cell B27: = sign(sin(A27))

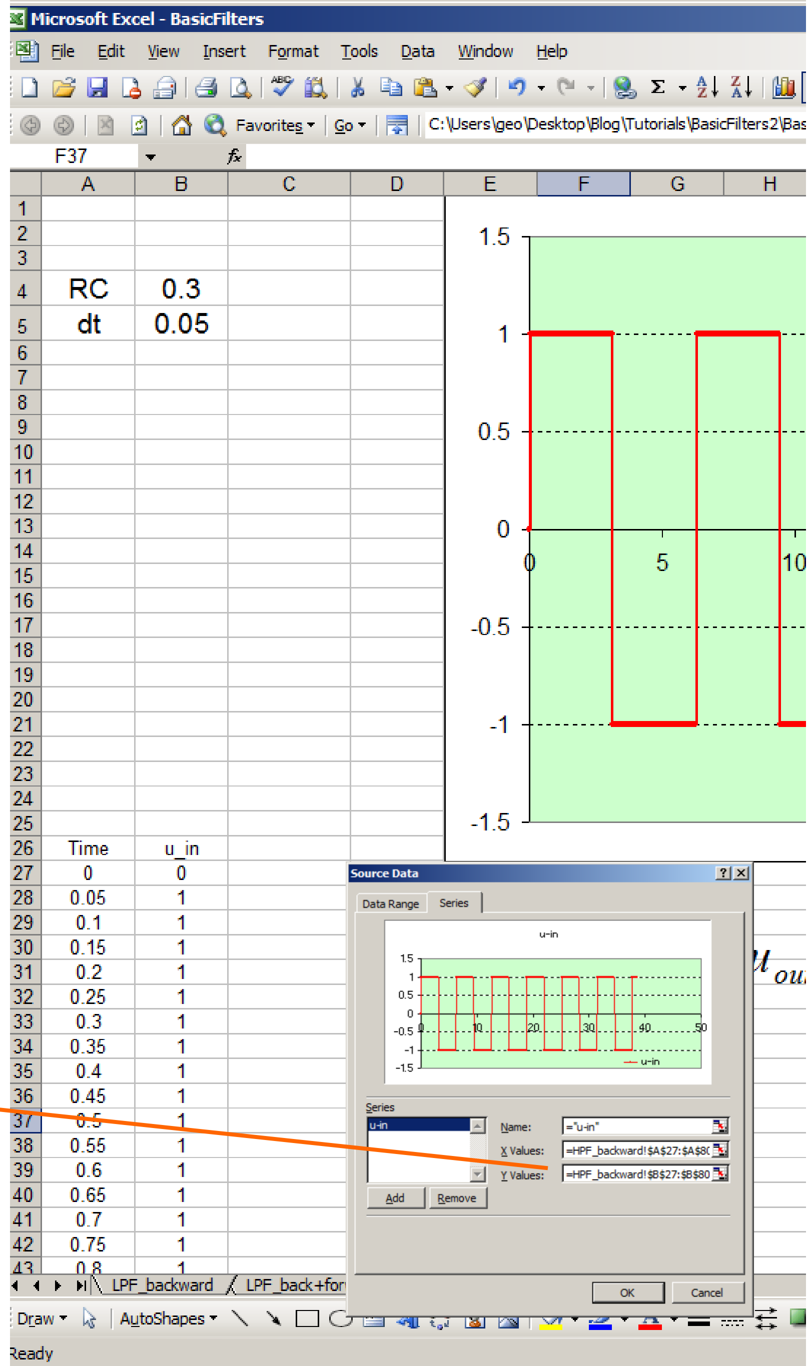
Copy B27 down to B800

(this will generate the input series on column B)

Plot the input signal:

Create a x-y scatter plot





Create an output series named “u-out-HPF”:

Cell C26: u_out_HPF (a label)

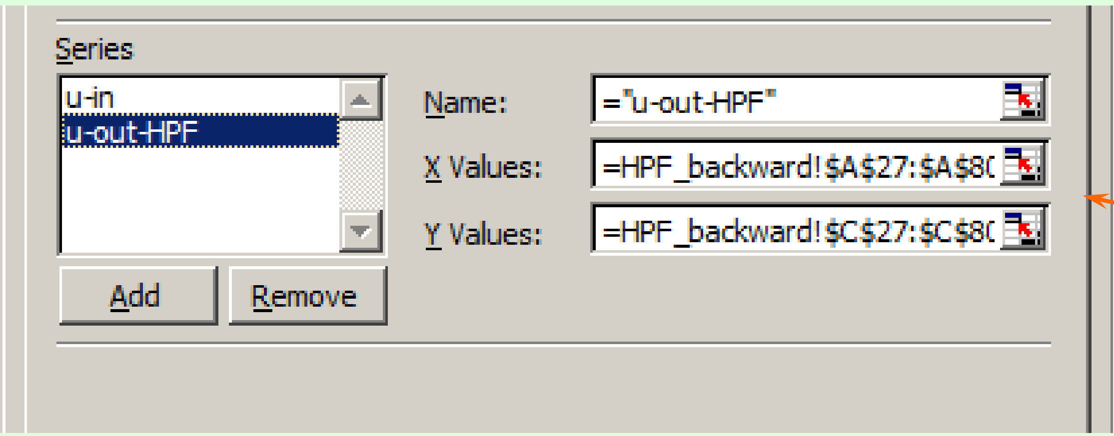
Cell C27: 0

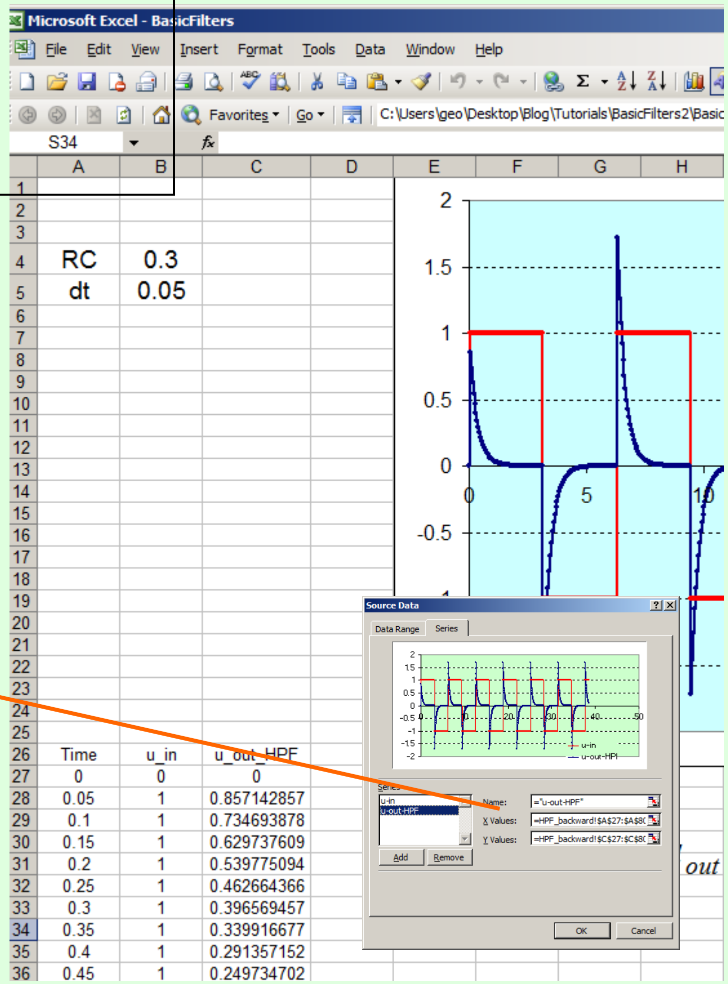
Cell C28: =(C27+B28-B27)/ (1+$B$5/$B$4)

Copy C27 down to C800

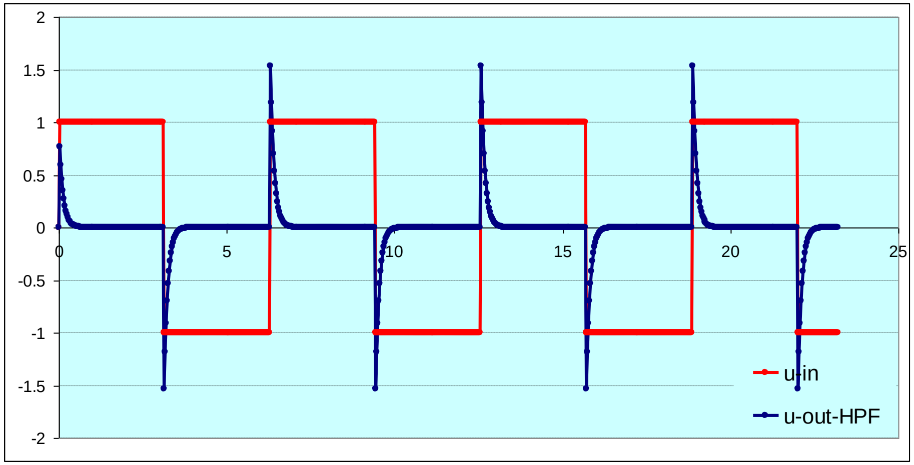
(this will generate the output series on column B)

Add the output series on the chart:

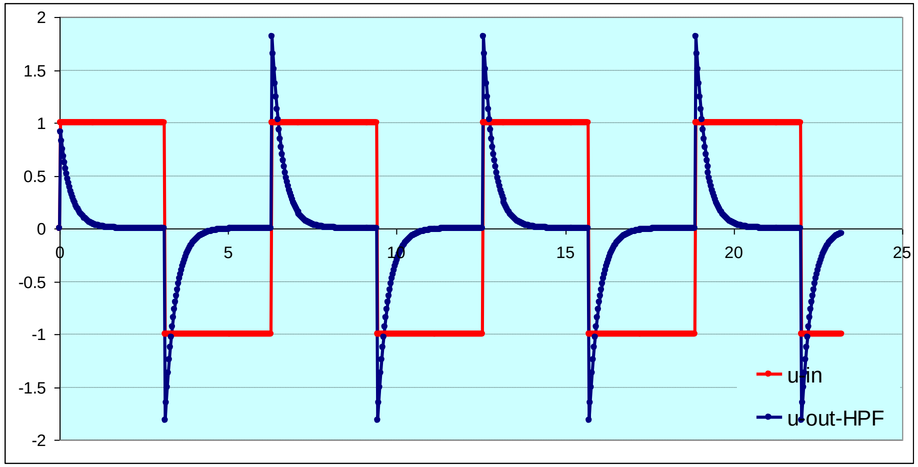




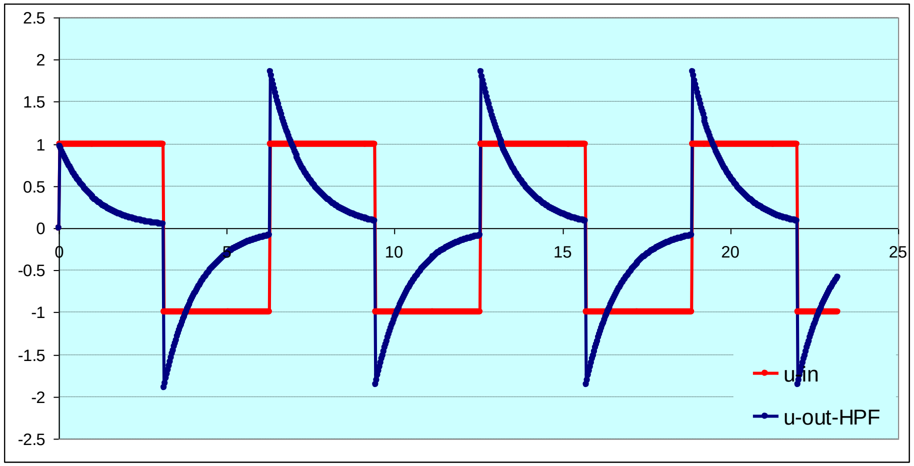
Few situations:



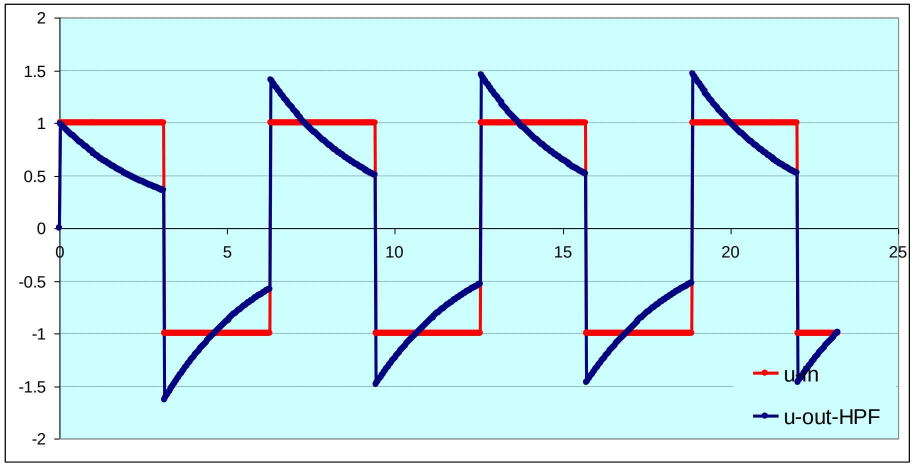
dt=0.03, RC=0.1



dt=0.03, RC=0.3



dt=0.03, RC=1



dt=0.03, RC=3

Let's try to use ***forward estimate*** of the derivative instead of the ***backward estimate*** and see if we get the same numerical result:

$$u_{out}(t) = R \cdot C \cdot \left(\frac{du_{in}}{dt} - \frac{du_{out}}{dt} \right) \quad =>$$

Instead of this
(backward estimate)

$$h \cdot u_{out}(t) = R \cdot C \cdot \left(u_{in}(t) - u_{in}(t-h) - u_{out}(t) + u_{out}(t-h) \right)$$

Use the this
(forward estimate)

$$h \cdot u_{out}(t) = R \cdot C \cdot \left(u_{in}(t+h) - u_{in}(t) - u_{out}(t+h) + u_{out}(t) \right)$$

After some algebraic manipulation we reach the final formula for $u_{out}(t)$:

$$u_{out}(t+h) = u_{out}(t) \cdot \left(1 - \frac{h}{R \cdot C}\right) + u_{in}(t+h) - u_{in}(t)$$

Let's add a new output column in the same Excel work book:

Create a copy of the worksheet "HPF_backward" and rename it "HPF_backward+forward"

Add a new out column:

Cell D26: U_out_f (a label)

Cell D27: 0

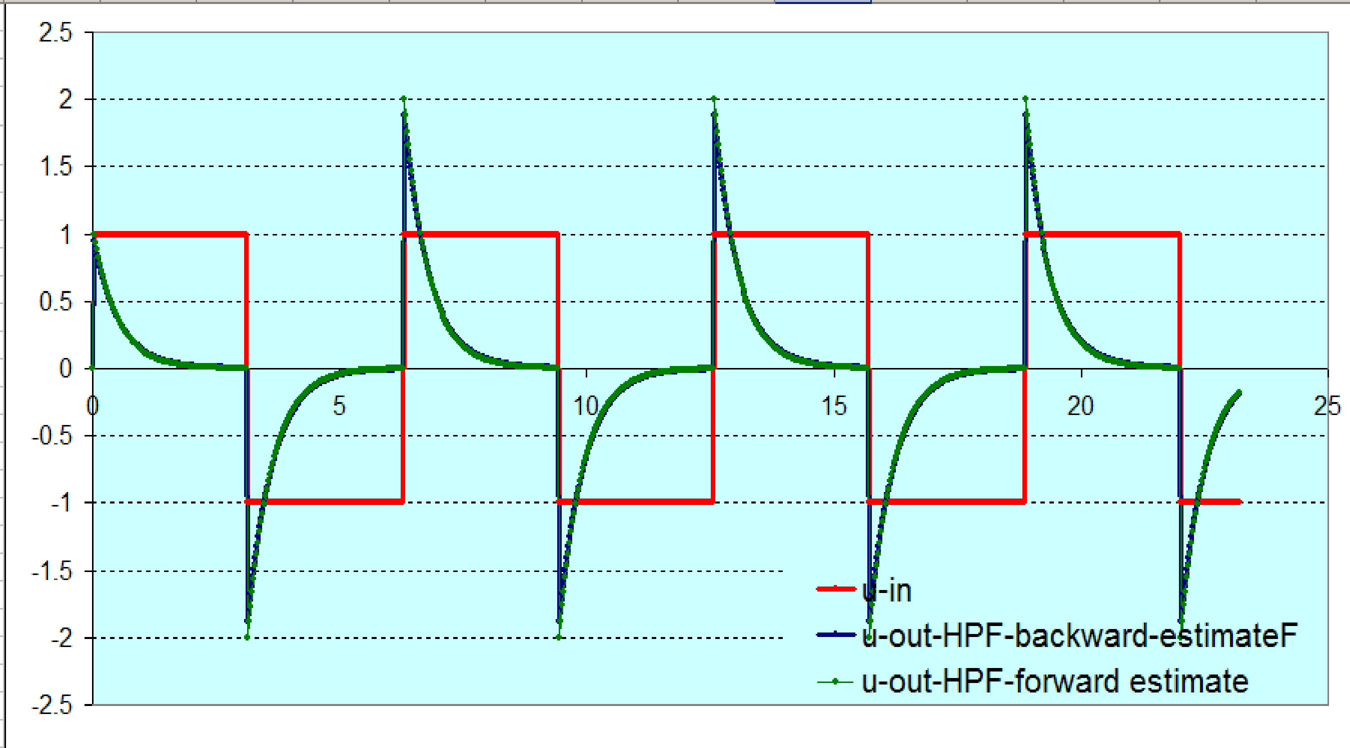
Cell D28: = D27*(1-$B$5/$B$4)+B28-B27

Copy C28 down to D800

Add column as a series to the chart named: "u-out-HPF_forward_estimate"

Microsoft Excel - BasicFilters

RC	0.5
dt	0.03



Time	u_in	u_out_HPF_b	u_out_HPF_f
0	0	0	0
0.03	1	0.943396226	1
0.06	1	0.88999644	0.94
0.09	1	0.839619283	0.8836
0.12	1	0.792093663	0.830584
0.15	1	0.747258173	0.78074896
0.18	1	0.70496054	0.733904022
0.21	1	0.665057114	0.689869781
0.24	1	0.627412371	0.648477594
0.27	1	0.591898464	0.609568939
0.3	1	0.558394777	0.572994802
0.33	1	0.526787525	0.538615114
0.36	1	0.496969364	0.506298207
0.39	1	0.468839022	0.475920315
0.42	1	0.442300964	0.447365096
0.45	1	0.417265061	0.42052319
0.48	1	0.393646284	0.395291799

- We can see that the two output waveforms overlap!
- It can be easily verified that this difference decreases by reducing the time step

LPF_backward / LPF_back+forward / HPF_backward / HPF_backward+forward