

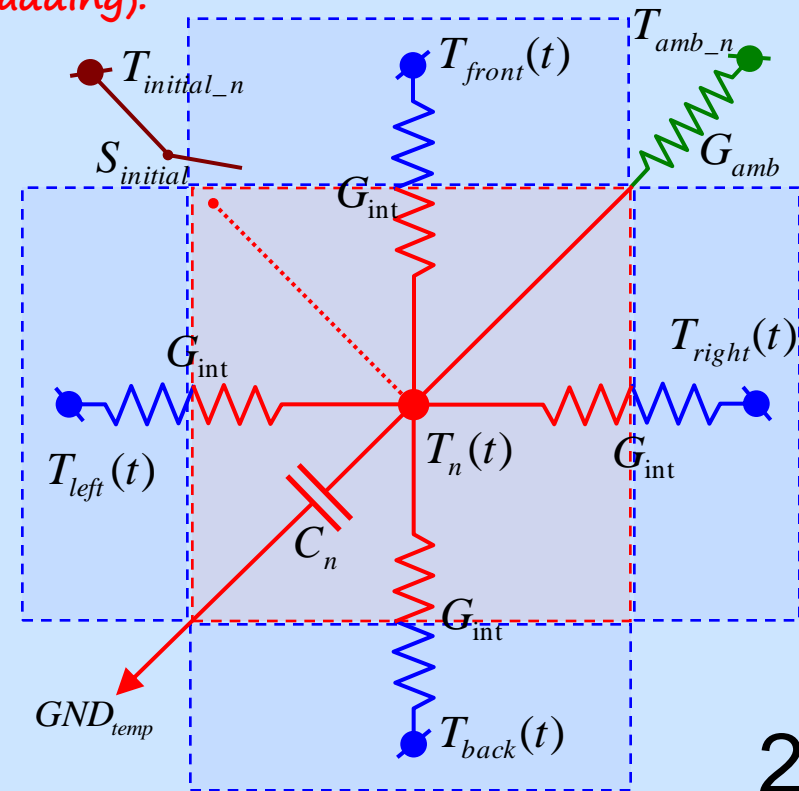
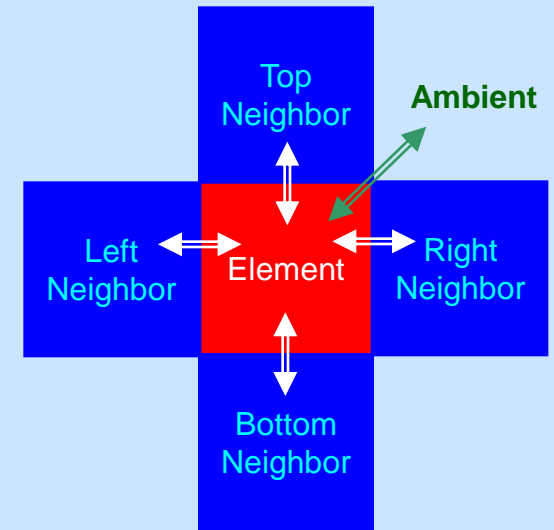
Building a Dynamic Two Dimensional Heat Transfer Model

– part #1

- This is the first half of a tutorial which shows how to build a basic dynamic heat conduction model of a square plate. The same principle could be used to model different shape 2D objects.
- This presentation explains how to partition the plate into elementary square sections each having four neighboring elements (right, front, left, back) and each element being characterized by three parameters: the heat capacitance, the heat conductance (between neighboring elements) and the heat conductance with the ambient environment.
- The section also explains how the heat transfer occurs between the elements by following the two rules of heat transfer: the storage and the transport equation. The final numerical temperature calculation formula is derived.

Partitioning the body and creating an equivalent element schematic:

- We divide the plate in equal squares. Each square has a heat (or thermal) capacitance C , a heat (thermal) conductance G_{int} and the conductance between each square (element) and the ambient environment is G_{amb} .
- Each element inside the plate has four immediate neighbors, the right, the front, the left and the back neighbor.
- In the calculations we give each element the same treatment. We will see that we can use the same formulas even on the edges and in the corners where an element has only three or two neighbors by applying some tricks (add fake neighbors by matrix padding).
- Each element changes heat with its immediate neighbors as well with the ambient environment.
- We use an electrical schematic because it is the easiest to draw and read.
- Notice that each element is modeled as a capacitor to ground storing heat (represented by temperature) instead of electric charge. The heat conductances with the neighbors are bi-color to represent the fact that they are shared between the neighbors.
- The switch $S_{initial}$ is being opened at the beginning of the simulation. At that point the heat capacitance C_n is already set to temperature $T_{initial_n}$.



Applying the storage and the transport equations:

$$\text{(storage equation)} \quad dQ = C \cdot dT \quad \text{(transport equation)} \quad \frac{dQ}{dt} = G \cdot (T_1 - T_2)$$

- Let's first write the transport equation for each of the five conductances

separately:

$$\left\{ \begin{array}{ll} \frac{dQ_{right}(t)}{dt} = G_{int} \cdot [T_{right}(t) - T_n(t)] & \text{- rate of heat transfer from the right neighbor} \\ \frac{dQ_{front}(t)}{dt} = G_{int} \cdot [T_{front}(t) - T_n(t)] & \text{- rate of heat transfer from the front neighbor} \\ \frac{dQ_{left}(t)}{dt} = G_{int} \cdot [T_{left}(t) - T_n(t)] & \text{- rate of heat transfer from the left neighbor} \\ \frac{dQ_{bottom}(t)}{dt} = G_{int} \cdot [T_{bottom}(t) - T_n(t)] & \text{- rate of heat transfer from the bottom neighbor} \\ \frac{dQ_{amb}(t)}{dt} = G_{amb} \cdot [T_{amb_n}(t) - T_n(t)] & \text{- rate of heat transfer from the ambient} \end{array} \right.$$

- Now let's write the storage equation for element n:

$$dQ_{total_n}(t) = dQ_{right}(t) + dQ_{front}(t) + dQ_{left}(t) + dQ_{back}(t) + dQ_{ambient_n}(t) = C \cdot dT_n(t)$$

- After some manipulation the above formula becomes:

$$G_{int} \cdot dt \cdot [T_{right}(t) + T_{front}(t) + T_{left}(t) + T_{back}(t) - 4 \cdot T_n(t)] + G_{amb} \cdot dt \cdot [T_{amb_n}(t) - T_n(t)] = C \cdot dT_n(t)$$

Let's write the previous equation in a numerical form:

From the previous page we have:

$$G_{\text{int}} \cdot dt \cdot [T_{\text{right}}(t) + T_{\text{front}}(t) + T_{\text{left}}(t) + T_{\text{back}}(t) - 4 \cdot T_n(t)] + G_{\text{amb}} \cdot dt \cdot [T_{\text{amb}_n}(t) - T_n(t)] = C \cdot dT_n(t)$$

By using the following notations: $t = m$, $t - dt = m - 1$, $dt = h$ the above equation becomes:

$$G_{\text{int}} \cdot h \cdot [T_{\text{right}}(m) + T_{\text{front}}(m) + T_{\text{left}}(m) + T_{\text{back}}(m) - 4 \cdot T_n(m)] + G_{\text{amb}} \cdot h \cdot [T_{\text{amb}_n}(m) - T_n(m)] = C \cdot [T_n(m) - T_n(m-1)]$$

- We are trying to obtain $T_n(m)$ and if we solve the equation above, $T_n(m)$ will be dependent on $T_{\text{right}}(m)$, $T_{\text{front}}(m)$, $T_{\text{left}}(m)$ and $T_{\text{back}}(m)$. At their turn each of the four neighboring temperatures are calculated function of $T_n(m)$. Therefore in the form above the $T_n(m)$ system of equations for all the component elements leads to a mess of circular references.

Solution:

- Instead of using a back estimate for the differential of temperature function we can use a forward estimate:

$$dT_n(m) = T(m) - T(m-1) \quad (\text{back estimate})$$

$$dT_n(m) = T(m+1) - T(m) \quad (\text{forward estimate})$$

Using the forward estimate the main temperature equation becomes:

$$G_{\text{int}} \cdot h \cdot [T_{\text{right}}(m) + T_{\text{front}}(m) + T_{\text{left}}(m) + T_{\text{back}}(m) - 4 \cdot T_n(m)] + G_{\text{amb}} \cdot h \cdot [T_{\text{amb}_n}(m) - T_n(m)] = C \cdot [T_n(m+1) - T_n(m)]$$

Solving for $T_n(m+1)$:

$$T_n(m+1) = T_n(m) + \frac{h \cdot G_{\text{int}}}{C} \cdot [T_{\text{right}}(m) + T_{\text{front}}(m) + T_{\text{left}}(m) + T_{\text{back}}(m) - 4 \cdot T_n(m)] + \frac{h \cdot G_{\text{amb}}}{C} \cdot [T_{\text{amb}_n}(m) - T_n(m)]$$

- Above we have the needed equation for finding the future temperature function of current temperature of the element itself and the current temperature values of the near neighbor elements.
- The equation can be written in a more suggestive way as:

$$T_{n_future} = T_{n_current} + [G_{\text{int}} \cdot (T_{\text{right_current}} + T_{\text{front_current}} + T_{\text{left_current}} + T_{\text{back_current}} - 4 \cdot T_{n_current}) + G_{\text{amb}} \cdot (T_{\text{amb}_n_current} - T_{n_current})] \cdot \frac{h}{C}$$

- The equation above refers to any two consecutive moments in time (future & current) separated by one time step "h". In order to be aligned with our previous models we will rewrite the equations as follows (I used "past" instead of "previous" to save space on the slide).

$$T_{n_current} = T_{n_past} + [G_{\text{int}} \cdot (T_{\text{right_past}} + T_{\text{front_past}} + T_{\text{left_past}} + T_{\text{back_past}} - 4 \cdot T_{n_past}) + G_{\text{amb}} \cdot (T_{\text{amb}_n_past} - T_{n_past})] \cdot \frac{h}{C}$$

- We can see the similarity to the 1D equation:

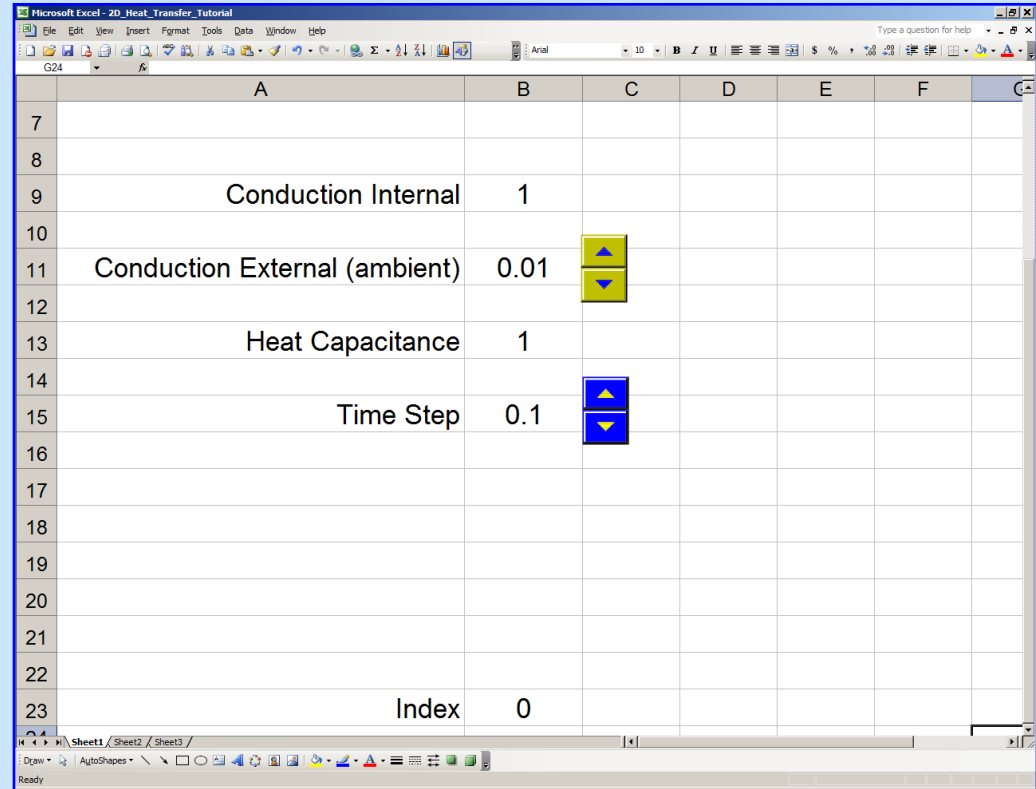
$$T_{n_current_1D} = T_{n_past} + [G_{\text{int}} \cdot (T_{\text{right_past}} + T_{\text{left_past}} - 2 \cdot T_{n_past}) + G_{\text{amb}} \cdot (T_{\text{amb}_n_past} - T_{n_past})] \cdot \frac{h}{C}$$

- In 3D, an element would have 6 neighbors instead of just four so the equation would become:

$$T_{n_current_2D} = T_{n_past} + [G_{\text{int}} \cdot (T_{\text{right_past}} + T_{\text{front_past}} + T_{\text{left_past}} + T_{\text{back_past}} + T_{\text{top_past}} + T_{\text{bottom_past}} - 6 \cdot T_{n_past}) + G_{\text{amb}} \cdot (T_{\text{amb}_n_past} - T_{n_past})] \cdot \frac{h}{C}$$

Excel implementation of the 2D conduction heat-transfer model:

- Open a new workbook and save it as "2D_Heat_Transfer_Tutorial"
- Insert the labels as seen in the snapshot to the right
- Using the "Control Toolbox" create two buttons and name them "ConductionToAmbient" and "Time_Step"
- Set the range of the first button (right click the "Properties" menu and change "Min" and "Max") to [0,100] and change the range of the second button to [1,100]
- After creating the buttons double click them while in design mode and the VBA editor will come up. Write the following code in the editor and save.
- After you finish make sure you exit the "Design Mode" before testing the functionality of the buttons.



```
Private Sub ConductionToAmbient_Change()  
    [B11] = ConductionToAmbient.Value / 100  
End Sub
```

```
Private Sub Time_Step_Change()  
    [B15] = Time_Step.Value / 500  
End Sub
```