Introduction to Geometrical Optics – a 2D ray tracing Excel model for spherical mirrors – Part 2

by George Lungu

– This is a tutorial explaining the creation of an exact 2D ray tracing model for both spherical concave and spherical convex mirrors.

– While the previous section dealt with creating and plotting the mirror, this section explains the geometry of the 25-ray light beam emerging from an artificial star 'L''.

– This is an exact model in the sense that no geometrical approximations are used, however the model does not take into consideration diffraction effects.

The input beam parameters:

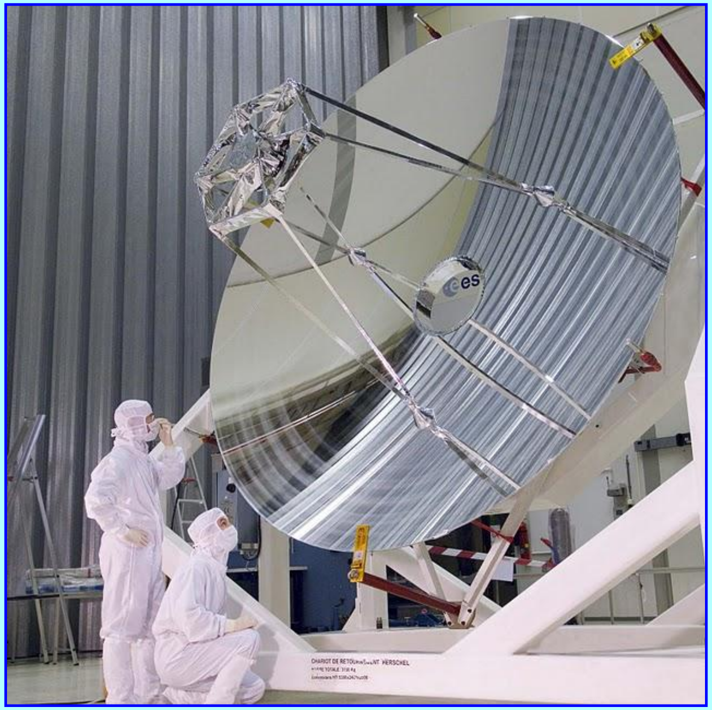
– There are two variable beam parameters set by the user: x_L and alpha. Parameter y_L will be calculated from x_L x_M, y_M and alpha.

– Essentially we have two points L and M and we write the slope formula of the segment connecting them (see the diagram in the next page):

$$\tan(alpha) = \frac{y_M - y_L}{x_M - x_L}$$

and from here we can derive the formula for y_L:

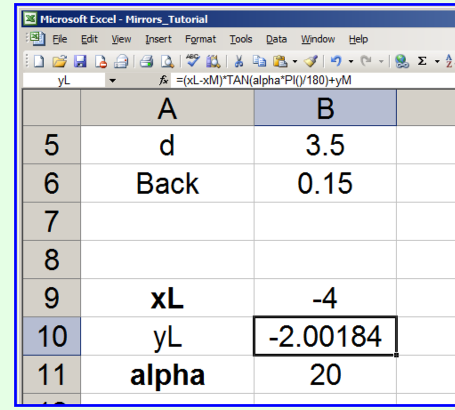
$$y_L = y_M + (x_L - x_M) \cdot \tan(alpha)$$



The European Space Agency's Herschel telescope mirror (Far InfraRed and SubMillimeter telescope - FIRST)

Create spreadsheet entries for the input beam parameters in the spreadsheet:

- Range A9:A1 contains labels
- Name call B9 "xL", name cell B10 "yL" and cell B1 "alpha"
- B10: "=(xL-xM)*TAN(alpha*PI()/180)+yM"
- xL and alpha are input parameters and you can type some constants in those cells.


Microsoft Excel - Mirrors_Tutorial

yL | =(xL-xM)*TAN(alpha*PI()/180)+yM

	A	B
5	d	3.5
6	Back	0.15
7		
8		
9	**xL**	-4
10	yL	-2.00184
11	**alpha**	20

Calculating the incidence parameters:

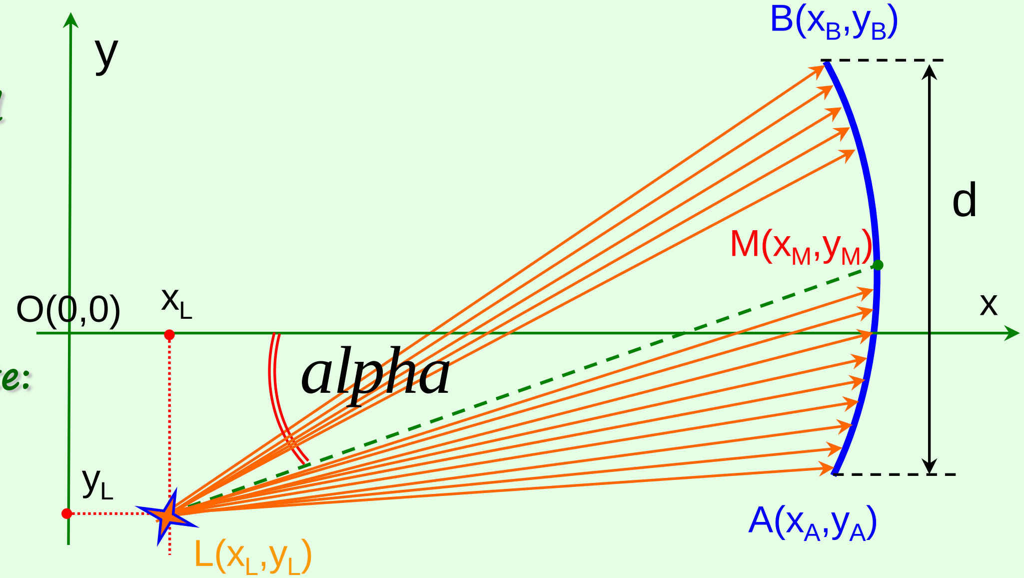
-Let's create 25 rays starting from point L with the condition that all hit the mirror and are uniformly spread.

- The first (lowest) ray will hit the mirror at the lowest extremity (A) and the last one at the highest extremity (B)

- Based on the above description we can write:

$$\begin{cases} \alpha_{\min} = \operatorname{atan}\left(\dfrac{y_A - y_L}{x_A - x_L}\right) \\ \alpha_{\max} = \operatorname{atan}\left(\dfrac{y_B - y_L}{x_B - x_L}\right) \end{cases}$$

We will retrieve the coordinates of points A and B from the mirror table where they were already calculated B42:C42 and B62:C62



Let's express the angle difference between two consecutive rays

$$\alpha_{delta} = \frac{\alpha_{\max} - \alpha_{\min}}{24}$$

Spreadsheet implementation of the incidence parameters:

- Range A13:A15 contains just labels.
- Name B13 "alpha_min", B14 "alpha_max" and B15 "delta_alpha"
- B13: "=ATAN((C42-yL)/(B42-xL))
- B14: "=ATAN((C62-yL)/(B62-xL))
- B15: "=(alpha_max-alpha_min)/24"

	A	B	C
7			
8			
9	**xL**	-4	
10	yL	-2.00184	
11	**alpha**	20	
12			
13	alpha_min	0.045756	
14	alpha_max	0.598647	
15	delta_alpha	0.027645	
16			

Writing the equations of the incident rays:

- We have 25 incident rays starting from the artificial star L. In the previous two pages we derived constraints for the angle of these rays so that they all end up meeting the mirror surface and covering it uniformly.
- Knowing one point through which all the rays pass (point L of coordinates (xL,yL)) and knowing the angle of each ray with respect to the horizontal x-axis we can write the following 25 equations:

$$\tan(\alpha_i) = \frac{y - y_L}{x - x_L} \qquad \text{for } i=0,1,2,\ldots\ldots, 24$$

we can write the previous expression as: $\Rightarrow y - y_L = (x - x_L)\cdot\tan(\alpha_i)$

And we can finally bring it to a standard line equation in the x-y plane:

$$\Rightarrow \boxed{y - \tan(\alpha_i)\cdot x + \left[x_L \cdot \tan(\alpha_i) - y_L\right] = 0 \quad \text{for } i=0,1,2,\ldots\ldots, 24}$$

Writing the equation of the mirror surface (it's actually a curve since it's all in 2D):

- We need the mirror surface equation so that forming a system with the previously derived incident ray equations we can calculate the exact coordinates of the points where the incidence rays meet the mirror. With that information we can also calculate the slope of the reflected rays hence being able to write the equations of the reflected rays and chart them.

- A standard circle equation in the x-y plane looks like this:

$$(x - x_C)^2 + (y - y_C)^2 = R^2$$

where (x_C, y_C) are the Cartesian coordinates of the origin

In our case the mirror equation becomes:

$$(x - x_M - R)^2 + (y - y_M)^2 = R^2$$

Which can be rewritten as:

$$\boxed{(x - x_M - R)^2 + (y - y_M)^2 = R^2}$$

Combining the incident ray equations with the mirror equation we get the following system to solve to find the incidence coordinates (intersection points between the rays and the mirror):

$$\begin{cases} (x - x_M - R)^2 + (y - y_M)^2 = R^2 \\ y - \tan(\alpha_i) \cdot x + [x_L \cdot \tan(\alpha_i) - y_L] = 0 \end{cases}$$

for i=0,1,2,....., 24

Solving the system of equations to find the incidence points:

We have the following system:

$$\begin{cases} (x - x_M - R)^2 + (y - y_M)^2 = R^2 \\ y = \tan(\alpha_i)\cdot x + y_L - x_L\cdot\tan(\alpha_i) \end{cases}$$

for i=0,1,2,....., 24

Let's substitute the y from the latter equation into the former equation and for now carry just the former equation:

$$(x - x_M - R)^2 + (x\cdot\tan(\alpha_i) + y_L - y_M - x_L\cdot\tan(\alpha_i))^2 = R^2$$

$$\left(1+\tan^2(\alpha_i)\right)\cdot x^2 + 2x\cdot\left[\tan(\alpha_i)\cdot\left(y_L - y_M - x_L\cdot\tan(\alpha_i)\right) - x_M - R\right] +$$
$$+ (x_M + R)^2 + (y_L - y_M - x_L\cdot\tan(\alpha_i))^2 - R^2 = 0$$

The equation becomes:

$$a\cdot x^2 + b\cdot x + c = 0$$

where a, b, c are given here:

$$\begin{cases} a = \dfrac{1}{\cos^2(\alpha_i)} \\ b = 2\cdot\left[\tan(\alpha_i)\cdot\left(y_L - y_M - x_L\cdot\tan(\alpha_i)\right) - x_M - R\right] \\ c = (x_M + R)^2 + (y_L - y_M - x_L\cdot\tan(\alpha_i))^2 - R^2 \end{cases}$$

$$a \cdot x^2 + b \cdot x + c = 0$$

This is a quadratic equation whose solution is:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$$

If the determinant: $\Delta = b^2 - 4 \cdot a \cdot c < 0$ the equation has no solution

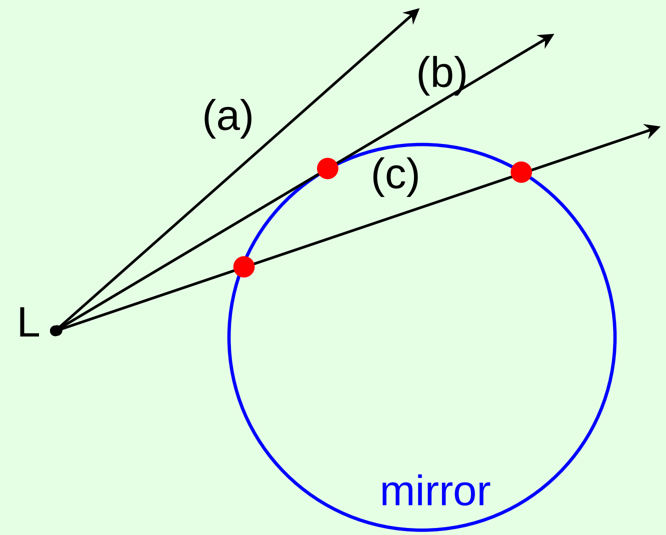
In most of cases there are two solutions since a line usually intersects a circle in two points.

Judging by the diagram to the left we can have three conditions:

(a) the ray misses the mirror (no solution to the quadratic equation - the determinant is negative)

(b) the ray is tangent to the mirror (the determinant is null).

(c) the ray intersects the mirror twice (the determinant is positive) and we need to choose the higher value (with +) for a concave mirror (R < 0) and choose the lower value (with minus in front of the square root) for convex mirrors (R > 0).



Of course in the spreadsheet we need to introduce one more condition in the function, namely when the ray goes higher than the upper end of the mirror or lower than the lower end of the mirror the ray misses the mirror and it passes un-deviated.

to be continued...